Comparison of Weibayes and Markov Chain Monte Carlo methods for the reliability analysis of turbine nozzle components with right censored data only

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Abstract

The Weibull distribution is widely used in reliability engineering to represent the component failure behaviour. The parameters of this distribution cannot be estimated by applying the widely used Maximum Likelihood Estimation (MLE) method when the collected field data contain right-censored times only. To overcome this limitation, the Weibayes method is often used in industrial practice: it consists in setting the value of the shape parameter based on prior knowledge and, then, estimating a lower confidence bound on the scale parameter. An alternative approach to estimate the Weibull parameters relies on the Markov Chain Monte Carlo technique, within the Bayesian statistics framework. This technique allows accommodating poor information on the parameter values, which is modeled by vague prior distributions. In this paper, a comparison between the Weibayes and MCMC approaches is proposed by way of a real industrial case study concerning data on Gas Turbine (GT) forced outages due to the mechanical failure of a GT component.

Key Words: Weibayes; Bayesian Analysis; Markov Chain Monte Carlo (MCMC);

1 INTRODUCTION

The Weibull distribution is probably the most widely used distribution in reliability engineering (Crowder et al., 1991). The reasons for this popularity is meanly due to the variety of shapes, and thus failure behaviors, it can accommodate.

In general, both the scale and shape parameters of the Weibull distribution can be easily estimated by means of the Maximum Likelihood Estimation (MLE) method, as failure time datasets usually contain both actual failure times and right-censored observations (i.e., no failure occurred during the test time). However, when all the collected data are right-censored times, MLE can no longer be applied (Di Maio et al., 2015). To overcome this limitation, the Weibayes approach has been recently introduced (Abernethy, 2008) and adopted by some widely used commercial software tools for reliability engineering.

The Weibayes method asks the experts to give a precise (i.e., with no uncertainty) value of the shape parameter based on prior experience or knowledge. Then, it provides a lower confidence bound on the scale parameter. Such precise estimation of the shape parameter may yield misleading results, if it is wrong.

The Bayesian paradigm offers an alternative approach to estimate the parameters of the Weibull distribution, which are framed as random variables with their own probability distributions, so called prior distributions. Poor information on the parameter values can be accommodated by vague prior distributions and the Bayesian inference procedure allows adjusting them based on the evidence coming from field data, also reducing the uncertainty of the initial prior distributions. The posterior distribution thereby obtained encodes both the prior knowledge of the expert and the statistical evidence collected, and the posterior mean, median and credible intervals of the failure time can be extracted.

However, analytical approaches to derive the posterior distributions are not always feasible in practical cases, such as that of Weibull components with no failure experienced during the test time and with only non-informative priors available. In these cases, one can resort to the Markov Chain Monte Carlo (MCMC) algorithm, which is an advanced Monte Carlo sampling technique that requires specific theoretic knowledge and experience for its use (Robert & Casella, 2004).

In this work, Weibayes and Bayesian approaches are compared by way of a case study concerning data on Gas Turbine (GT) forced outage due to the mechanical failure of a GT component. The case study considered in this work is derived from a real industrial application. However, the details on both the component and the degradation mechanism that lead it to failure are not given to protect the intellectual property of General Electric (GE).

The paper is organized as follows: in Section 2, basics of reliability analysis based on Weibull distribution are recalled; in Section 3, the Weibayes method is discussed. Section 4 presents Weibull Bayesian analysis for dealing with right-censored data only. The application of the methodology to the GE's case study is described in Section 5, whereas in Section 6 conclusions are drawn.

2 WEIBULL RELIABILITY ANALYSIS

In this work, we assume a Weibull Probability Density Function (pdf), $f(\tau)$, for the stochastic failure time *T*:

$$f(\tau)=\frac{\beta}{\alpha}\left(\frac{\tau}{\alpha}\right)^{\beta-1}e^{-\left(\frac{\tau}{\alpha}\right)^{\beta}}\ \tau>0,\ \alpha>0,\ \beta>0 \tag{1}$$

where β is the shape parameter and α is the scale parameter of the distribution. The corresponding reliability function is given by:

$$R(\tau)=e^{-\left(\frac{\tau}{\alpha}\right)^{\beta}}\ \tau>0,\ \alpha>0,\ \beta>0 \tag{2}$$

whereas the hazard rate is:

$$h(\tau)=\frac{\beta}{\alpha}\left(\frac{\tau}{\alpha}\right)^{\beta-1}\ \tau>0,\ \alpha>0,\ \beta>0 \tag{3}$$

The shape parameter can be interpreted as follows:

- β <1 corresponds to a decreasing hazard rate behavior. This models the behavior of components for which the failure frequency is larger when they are put into service and decreases over time.
- β =1 in this case, the hazard rate is constant, and the Weibull distribution reduces to an exponential distribution.
- β >1 corresponds to an increasing hazard rate behavior, which is typical of aging components.

Let Γ be a positive random variable denoting the time at which a right-censoring kicks in: we observe either failure time *T* or censoring time Γ, whichever comes first. It follows that the observed dataset is a collection of random variables (Christensen et al., 2010):

$$Y=\min(T,\Gamma) \tag{4}$$

In addition, the information on whether *Y* is an actual failure time or a censored observation is known, and is modeled by the indicator variable

$$\delta=\begin{cases}0 & T\le\Gamma\\ 1 & T>\Gamma\end{cases} \tag{5}$$

That is, δ is set to 1 if we observe an actual failure time, otherwise it is set to 0. We assume that *T* and Γ are statistically independent random variable, and that the censoring distribution does not depend on parameters α and β. These conditions are usually referred to as non-informative censoring (Christensen et al., 2010).

In practice, the parameters of $f(\tau)$ are unknown and need to be estimated from data. To do this, let (y_i, δ_i) be independent identically distributed observations on i=1,...,n units. For convenience, the first k observations are failure times and the remaining n-k observations are right-censored outcomes. Then, the likelihood of data $\boldsymbol{D}$=($\boldsymbol{y}$,$\boldsymbol{\delta}$), where $\boldsymbol{y}$=(y_1,…,y_n) and $\boldsymbol{\delta}$=(δ_1,…,δ_n), reads (Christensen et al., 2010):

$$L(\alpha,\beta \mid \boldsymbol{D}) \propto \prod [f(y_i \mid \alpha,\beta)]^{\delta_i}[1-R(y_i \mid \alpha,\beta)]^{1-\delta_i} \tag{6}$$

or, equivalently:

$$L(\alpha,\beta \mid \boldsymbol{D}) \propto \prod_{i=1}^{n} [h(y_i \mid \alpha,\beta)]^{\delta_i} R(y_i \mid \alpha,\beta) \tag{7}$$

The MLE technique estimates parameters α and β by maximizing Eqs. (6) or (7), or their the logarithm. In details, the ML estimates $\hat{\alpha}$ and $\hat{\beta}$ of parameters α and β, respectively, are obtained by solving the following nonlinear equations (Gonzalez-Gonzalez et al., 2014):

$$\frac{\sum_{i=1}^{n} y_i^{\hat{\beta}} \log(y_i)}{\sum_{i=1}^{n} y_i^{\hat{\beta}}} - \frac{1}{k}\sum_{i=1}^{k}\log(y_i) - \frac{1}{\hat{\beta}} = 0 \tag{8}$$

$$\hat{\alpha} = \left(\frac{\sum_{i=1}^{n} y_i^{\hat{\beta}}}{k} \right)^{\frac{1}{\hat{\beta}}} \tag{9}$$

When the observations are all right-censored (i.e., no failure occurred), Eq. (9) becomes:

$$L(\alpha,\beta \mid \boldsymbol{D}) \propto \prod_{i=1}^{n} R(y_i \mid \alpha,\beta) = e^{-\frac{\sum_{i=1}^{n} y_i^{\beta}}{\alpha^{\beta}}} \tag{10}$$

and the MLEs of parameters α and β no longer exist.

To overcome this problem, the Weibayes method has been introduced (Abernethy, 2008), which assumes that the shape parameter β is known from either prior experience or engineering knowledge on the physics of the failure.

Basically, if (y_1,…,y_n) is a set of samples drawn from the Weibull distribution in Eq. (2), then $\left(y_1^{\beta}, \ldots, y_n^{\beta}\right)$ can be regarded as samples from an exponential distribution of mean time to failure $\psi = \alpha^{\beta}$, where β is known. Now, recall that when the observations from an exponential distributions are all right-censored, then a one-sided, lower 100($1-\varepsilon$)% confidence bound on ψ is given by (Zio, 2007):

$$\psi \geq \frac{\sum_{i=1}^{n} y_i^{\beta}}{-\log(\varepsilon)} \tag{11}$$

Therefore, a one-sided, lower 100($1-\varepsilon$)% confidence bound on the scale parameter is given by:

$$\alpha \geq \left(\frac{\sum_{i=1}^{n} y_i^{\beta}}{-\log(\varepsilon)} \right)^{\frac{1}{\beta}} \tag{12}$$

The value of α corresponding to ε=0.37 (i.e., 63% confidence level), $\hat{\alpha}_{Weib}^{0.63}$, is the estimate that some commercial software tools provide in output to the reliability engineers. This value enters the maintenance decision process.

3 BAYESIAN ANALYSIS

Within the Bayesian paradigm, both scale parameter α and shape parameter β are positive random variables. The prior knowledge on their variability is specified in a joint prior distribution with pdf $\pi(\alpha,\beta)$. Information brought by dataset $\boldsymbol{D}$ is combined with $\pi(\alpha,\beta)$ by means of the Bayes formula:

$$\pi(\alpha,\beta|\boldsymbol{D}) = \frac{L(\alpha,\beta|\boldsymbol{D})\pi(\alpha,\beta)}{\int_0^{\infty}\int_0^{\infty} L(\alpha,\beta|\boldsymbol{D})\pi(\alpha,\beta)\,d\alpha\,d\beta} \quad \alpha>0,\beta>0 \tag{13}$$

where the conditional pdf $\pi(\alpha,\beta\,|\,\mathbf{D})$ is usually called posterior pdf (Robert & Casella, 2004). As usual, in this work we assume that random variables α and β are statistically independent. This implies that the prior pdf $\pi(\alpha,\beta)$ is the product of the marginal prior pdfs $\pi(\alpha)$ and $\pi(\beta)$ of α and β, respectively. That is:

$$\pi(\alpha,\beta) = \pi(\alpha)\pi(\beta) \quad \alpha>0,\beta>0 \tag{14}$$

When poor information is available, non-informative prior distributions can be elicited for both parameters as, for example, the improper extended Jeffery's prior (Al-Kutubi & Hibrahim, 2009). The Bayes formula in Eq. (13) with likelihood given by Eq. (10) and priors by Eq. (14) gives a posterior distribution which is proportional to:

$$\pi(\alpha,\beta\,|\,\mathbf{D}) \propto e^{-\frac{\sum_{i=1}^{n} y^{\beta}}{\alpha^{\beta}}} \pi(\alpha)\pi(\beta) \tag{15}$$

Eq. (15) defines the kernel of an unknown pdf. Thus, a Markov Chain Monte Carlo (MCMC) algorithm (Casella & Berger, 2004) can be exploited in order to obtain samples from the posterior distribution, which will be used to make posterior inference on parameters α and β.

3.1 ***Markov Chain Monte Carlo***

MCMC is a family of algorithms that allow drawing samples from a probability distribution $g(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ (usually referred to as target distribution), which are produced by an ergodic Markov chain $\{X_t\}_{t\geq 0}$ (Andrieu & Thoms, 2008).

The main building block of this class of algorithms is the Metropolis-Hastings (MH) algorithm. It requires the definition of a family of proposal distribution $\{q(\boldsymbol{\theta},\cdot), \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$, which generate possible transitions for the Markov chain, say from $\boldsymbol{\theta}$ to $\boldsymbol{\theta}'$. The transitions are accepted or rejected according to the probability

$$r(\boldsymbol{\theta},\boldsymbol{\theta}') = \min\left\{1, \frac{g(\boldsymbol{\theta})q(\boldsymbol{\theta}',\boldsymbol{\theta})}{g(\boldsymbol{\theta}')q(\boldsymbol{\theta},\boldsymbol{\theta}')}\right\} \tag{16}$$

Here, we focus on the (symmetric) random walk MH algorithm, in which $q(\boldsymbol{\theta},\boldsymbol{\theta}') = q(|\boldsymbol{\theta}-\boldsymbol{\theta}'|)$ for some symmetric probability density q on $\boldsymbol{\Theta}$. In this case, Eq. (16) reads:

$$r(\boldsymbol{\theta},\boldsymbol{\theta}') = \min\left\{1, \frac{g(\boldsymbol{\theta})}{g(\boldsymbol{\theta}')}\right\} \tag{17}$$

As proposal distribution, we choose the multivariate Gaussian distribution $Norm(\boldsymbol{\theta},\boldsymbol{\mu},\boldsymbol{\Sigma})$ with vector mean $\boldsymbol{\mu}=\boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma}$. This latter is a symmetric $2x2$ matrix, with 3 parameters to be set. The choice of their values is critical for the convergence of the MH algorithm: large values of standard deviations improve the effectiveness of the chain in spanning throughout $\boldsymbol{\Theta}$, but with small efficiency (i.e., large number of rejected samples). For this, algorithms that adaptively tune the parameters of $\boldsymbol{\Sigma}$ have been devised. In this work, we have used the Adaptive Random Walk Metropolis-Hastings (ARWMH) algorithm (Andrieu & Thomas, 2008).

The pseudo-code of this algorithm is briefly reported in the particular case under study, where we are interested in drawing from the joint posterior distribution of α and β; that is, in our case $\boldsymbol{\theta}=(\alpha,\beta)$, $g(\cdot)=\pi(\cdot\,|\,\boldsymbol{D})$, and $q(\cdot)$ is a bivariate Gaussian distribution (Andrieu & Thomas, 2008):

1. Initialize $\alpha_0, \beta_0, \boldsymbol{\rho}_0$ and $\boldsymbol{\Sigma}_0$, where α_0 and β_0 are the initial values for parameter α and β respectively; $\boldsymbol{\rho}_0$ is an initial value for parameter $\boldsymbol{\rho}_t$, $t \geq 0$, which enters the updating step of covariance matrix $\boldsymbol{\Sigma}_{t+1}$ in Eq. (20). $\boldsymbol{\Sigma}_0$ is the assigned starting value of covariance matrix of the proposal bivariate normal density $q(\cdot)$; ξ_0 is a user-valued constant whose value must be set within the interval $(\frac{1}{1+\lambda}, 1]$, where λ is set equal to $\frac{2.38^2}{p}$, and $p = 2$ is the number of parameters of the kernel (Andrieu & Thoms, 2008). This quantity enters the definition of the step-sizes $\{\gamma_t\}_{t\geq 0}$ in Eq. (18).

At iteration $t+1$, given $\alpha_t, \beta_t, \boldsymbol{\rho}_t$ and $\boldsymbol{\Sigma}_t$:

2. Sample $(\alpha', \beta') \sim Norm\left(\begin{bmatrix}\alpha' \\ \beta'\end{bmatrix}, \begin{bmatrix}\alpha_t \\ \beta_t\end{bmatrix}, \lambda\boldsymbol{\Sigma}_t\right)$
3. Compute $r\left(\begin{bmatrix}\alpha_t \\ \beta_t\end{bmatrix}, \begin{bmatrix}\alpha' \\ \beta'\end{bmatrix}\right) = \min\left\{1, \frac{\pi(\alpha_t, \beta_t \mid \boldsymbol{D})}{\pi(\alpha', \beta' \mid \boldsymbol{D})}\right\}$
4. Sample $u \sim Unif(0,1)$
5. Set $(\alpha_{t+1}, \beta_{t+1}) = \begin{cases} (\alpha', \beta') & if\ u \leq r \\ (\alpha_t, \beta_t) & if\ u > r \end{cases}$
6. Update

$$\gamma_{t+1} = \frac{1}{(t+1)^{\xi_0}} \tag{18}$$

$$\boldsymbol{\rho}_{t+1} = \boldsymbol{\rho}_t + \gamma_{t+1}\left(\begin{bmatrix}\alpha_{t+1} \\ \beta_{t+1}\end{bmatrix} - \boldsymbol{\rho}_t\right) \tag{19}$$

$$\boldsymbol{\Sigma}_{t+1} = \boldsymbol{\Sigma}_t + \gamma_{t+1}\left(\left(\begin{bmatrix}\alpha_{t+1} \\ \beta_{t+1}\end{bmatrix} - \boldsymbol{\rho}_t\right)\left(\begin{bmatrix}\alpha_{t+1} \\ \beta_{t+1}\end{bmatrix} - \boldsymbol{\rho}_t\right)' - \boldsymbol{\Sigma}_t\right) \tag{20}$$

After M iterations of the ARWMH algorithm, we obtain two Markov chains, i.e., $\{\alpha_t\}_{t\in\{1,\ldots,M\}}$ and $\{\beta_t\}_{\in\{1,\ldots,M\}}$, which are drawn from the marginal posteriors $\pi(\alpha|\boldsymbol{D})$ and $\pi(\beta|\boldsymbol{D})$, of α and β, respectively.

To pointwise summarize the uncertainty in the posterior distributions, we consider the posterior median $\hat{\alpha}_{med}$:

$$\boldsymbol{P}\left(\alpha \le \hat{\alpha}_{med} \mid \boldsymbol{D}\right) = \int_0^{\hat{\alpha}_{med}} \pi(\alpha|\boldsymbol{D}) d\alpha = \frac{1}{2} \tag{21}$$

and the posterior mean $\hat{\alpha}_{mean}$

$$\hat{\alpha}_{mean} = \boldsymbol{E}\left\{\alpha|\boldsymbol{D}\right\} = \int_0^{\infty} \alpha \ \pi(\alpha|\boldsymbol{D}) d\alpha \tag{22}$$

Interval estimation of parameter α can also be given in terms of the $(1-\varepsilon)100\%$ Credible Interval (CI), which is the smallest subset $CI_{\alpha}^{1-\varepsilon} = \left(c_{inf}^{\alpha,1-\varepsilon}, c_{\sup}^{\alpha,1-\varepsilon}\right)$ of $\boldsymbol{R}_0^+$ such that:

$$\boldsymbol{P}(\alpha \in \left(c_{inf}^{\alpha,1-\varepsilon}, c_{\sup}^{\alpha,1-\varepsilon}\right)|\boldsymbol{D}) = \int_{c_{inf}^{\alpha,1-\varepsilon}}^{c_{\sup}^{\alpha,1-\varepsilon}} \pi(\alpha|\boldsymbol{D}) d\alpha = 1-\varepsilon \ , \varepsilon \in (0,1) \tag{23}$$

In particular, if we set in Eq. (23) $c_{\sup}^{\alpha,1-\varepsilon} \equiv +\infty$, then the value $c_{inf}^{\alpha,1-\varepsilon}$ that solves Eq. (23) is the is the lower bound $c_{Lower}^{\alpha,1-\varepsilon}$ of the one-sided $(1-\varepsilon)100\%$ lower credible interval $CI_{\alpha}^{1-\varepsilon}$. Mutatis mutandis, the definitions of posterior mean and median, and credibility interval given for the scale parameter α are the same for parameter β.

4 CASE STUDY

In this Section, we illustrate the application of the proposed methods to the GT forced outage due to a mechanical failure of GT component, which is assumed obeying a Weibull distribution. We rely on a dataset $\boldsymbol{D}$ containing $n = 20$ right-censored observations. For confidentiality, these values are not reported.

To clearly highlight the difference between the two methods, we consider two situations:

1. Realistic problem: in this case, the value of the shape parameter currently used by GE is considered.
2. Biased problem: in this case, the shape parameter is set to a value vary far from that used by GE.

4.1 *Realistic Problem*

According to the GE practice, the value of the shape parameter is set to

$$\hat{\beta}_{Weib} = 6 \tag{24}$$

This value is derived from thorough engineering considerations, not reported for confidentiality. The application of the Weibayes approach when considering the one-sided 63% lower confidence bound on the scale parameter yields

$$\alpha_{Weib}^{0.63} \approx 662 \quad (25)$$

The MCMC bayesian analysis has been performed with the following prior distributions:

- The generalized improper Jeffery prior distribution

$$\pi(\alpha) \propto \frac{1}{\alpha^{a}}, \alpha > 0, a > 0 \quad (26)$$

  with hyper-parameter $a = 2$. This corresponds to a diffuse prior distribution (i.e., with a wide support on $\mathbb{R}^{+}$), whose large variability is coherent with our poor knowledge base on its actual value.
- For the shape parameter, we have chosen a distribution centered on β=6, with probability mass of 0.8 uniformly distributed between [5.8,6.2] (i.e., symmetrically on β=6), a probability mass of 0.1 uniformly distributed in [0,5.8], whereas the remaining probability mass of 0.1 is uniformly distributed in [6.2,8]. This choice is justified by the following considerations:
  - To exploit the GE prior knowledge we have put a large portion of probability on the interval [5.8, 6.2] that encompasses the value $\hat{\beta}_{Weib} = 6$ provided by sound engineering considerations. In fact, we expect that if this value is correct, the MCMC for shape parameter will result in a Markov chain moving not too far from this value.
  - The prior distribution leaves a relatively small portion of probability in the remaining parts of the interval [0,8]. This is consistent with our prior knowledge: we assume that values in [0,8] are plausible values for parameter β.
  - Values of scale parameters larger than 8 correspond to very small uncertainty in the failure times (values of standard deviation almost 13% of the scale parameter). This situation is not realistic.

To draw samples from the posterior distribution $\pi(\alpha,\beta|\boldsymbol{D})$, the ARWMH algorithm described in Section 4 has been run for M=5,000,000 iterations with variables $\alpha_0, \beta_0, \boldsymbol{\rho}_0$ $\boldsymbol{\Sigma}_0$, and ξ_0 initialized as reported in Table 1.

α_0	β_0	$\boldsymbol{\rho}_0$	$\boldsymbol{\Sigma}_0$	ξ_0

800	3	$\begin{pmatrix} 1800 \\ 6 \end{pmatrix}$	$\begin{pmatrix} 120000 & -10 \\ -10 & 0.1 \end{pmatrix}$	0.4457

Table 1. Values of parameter $\alpha_0, \beta_0, \rho_0$ $\mathbf{\Sigma}_0$ and ξ_0

Then, we have applied:

1. a burn-in of 2,000,000 samples, i.e., the first 2,000,000 samples have been discarded to eliminate the bias introduced by the position of the initial point.
2. a sub-sampling (commonly referred to as thinning) every 100 samples to reduce the correlation between the successive points of the Markov chains generated by the algorithm.

By so doing, the cardinality of the original Markov chains has reduced from 3,000,000 to $\tilde{M} = 30000$ sample points. The trace plots (i.e., the plot of the sampled point, ordinate, vs the sample step, abscissa) relevant to the Markov chains $\{\alpha_t\}_{t \in \{1,\ldots,\tilde{M}\}}$ and $\{\beta_t\}_{t \in \{1,\ldots,\tilde{M}\}}$ are shown in Figures 1 and 2, respectively. From these Figures, it emerges that there is good mixing, i.e., the domains of the two posterior distributions $\pi(\alpha|\boldsymbol{D})$ and $\pi(\beta|\boldsymbol{D})$ are well explored around the distribution modes. In particular, the trace plot of $\{\beta_t\}_{t \in \{1,\ldots,\tilde{M}\}}$ in Figure 2 shows that this chain tends to sample in proximity of $\hat{\beta}_{Weib} = 6$, although samples are rarely (i.e., with small probability) drawn also from the remaining part of the support.

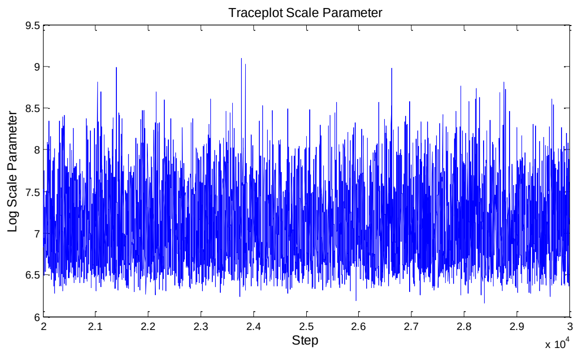

Figure 1: trace plot of Markov chain $\{\alpha_t\}_{t \in \{1,\ldots,\tilde{M}\}}$ (last 10000 sample points).

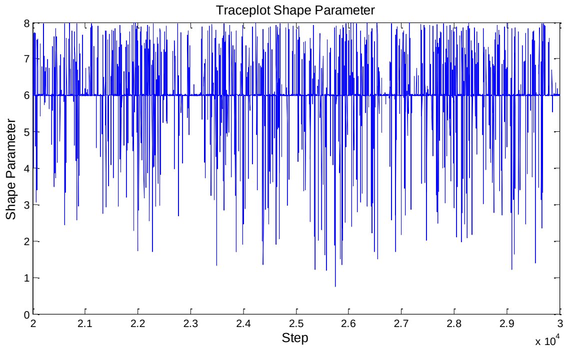

Figure 2: trace plot of Markov chain $\{\beta_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ (last 10000 sample points).

Figures 3 and 4 show the autocorrelation plots of chain $\{\alpha_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ and $\{\beta_t\}_{t\in\{1,\ldots,\tilde{M}\}}$, respectively. That is, for $l=1,2,\ldots$, 40 we measure the extent to which the values of the chain at time $(t+l)$ and time t are linearly related, for every $t=1,\ldots,\ \tilde{M}$.

From these Figures, it results that samples from both Markov chains can be considered almost uncorrelated.

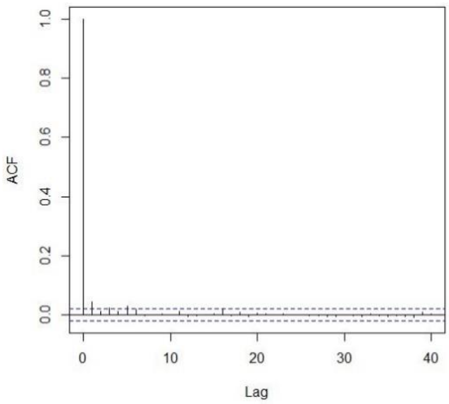

Figure 3: Autocorrelation plot of Markov chain $\{\alpha_t\}_{t\in\{1,\ldots,\tilde{M}\}}$

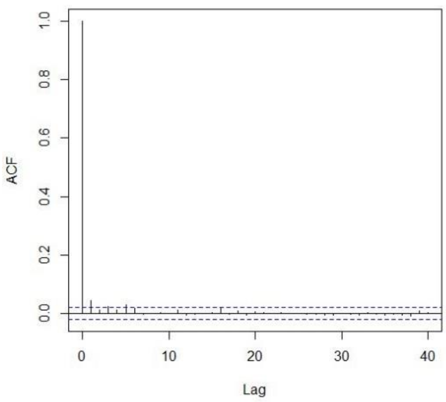

Figure 4: Autocorrelation plot of Markov chain $\{\boldsymbol{\beta}_t\}_{t\in\{1,\ldots,\tilde{M}\}}$

To assess the convergence to the posterior distribution, i.e., the stationarity of the two Markov chains $\{\alpha_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ and $\{\beta_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ we have exploited two standard diagnostic methods, whose results are summarized in Table 2:

- the Effective Sample Size (ESS), which gives an estimate of the equivalent number of independent iterations that the chain represents (Robert & Casella, 2010). For example, the 30000 samples from chain $\{\alpha_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ contain 11682 independent samples (Table 2, second column, second row), being the information in the remaining 18318 samples already contained in those 11682.
- The Geweke Test takes two nonoverlapping parts (usually the first 0.1 and last 0.5 proportions) of the Markov chain and compares the means of both parts, using a difference of means test to see if the two parts, of the chain are from the same distribution (null hypothesis) (Robert & Casella, 2010).

From Table 2, it emerges that the algorithm has converged to the desired target distribution.

Diagnostic Method			Comments
ESS	11681	29981	Passed
Geweke Test p-value	0.333	0.358	The null hypothesis of stationarity cannot be refused

Table 2: Results of some diagnostic methods to assess stationarity of the two Markov chains $\{\alpha_t\}_{t\in\{1,\ldots,\tilde{M}\}}$ and $\{\beta_t\}_{t\in\{1,\ldots,\tilde{M}\}}$

Figures 5 and 6 show the Kernel Density Estimation (KDE) (continuous line) and the Empirical Histogram (EH) of the estimated posterior pdfs $\pi(\alpha|\boldsymbol{D})$ and $\pi(\beta|\boldsymbol{D})$, respectively. The information in the posterior distribution is summarized by the values of posterior mean and median (Eqs. (21-22), respectively), 95% CI, and one-sided 63% lower credibility bound for scale parameters. These are reported in Table 3, whereas Table 4 reports those of the shape parameter.

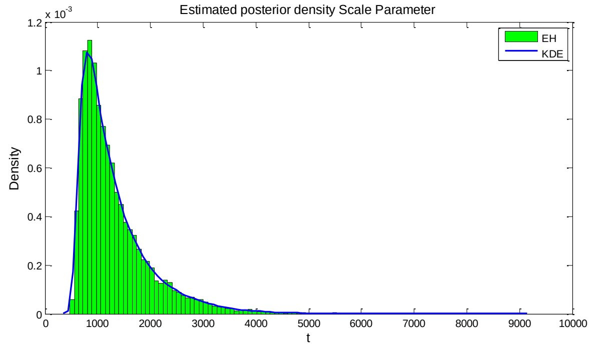

Figure 5: KDE and EH of posterior pdf $\pi(\alpha|\boldsymbol{D})$

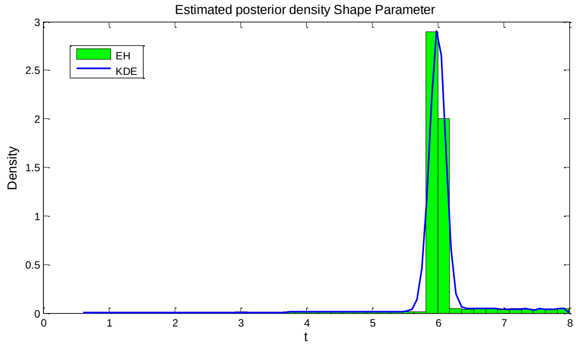

Figure 6: KDE and EH of posterior pdf $\pi(\beta|\boldsymbol{D})$

$\hat{\alpha}_{mean}$	$\hat{\alpha}_{med}$	$CI_{\alpha}^{0.95}=\left(c_{inf}^{\alpha,0.95},c_{\sup}^{\alpha,0.95}\right)$	$c_{Lower}^{\alpha,0.63}$
1321.5	1110.9	(615.7, 3170)	921.7

Table 3: Posterior mean ($\hat{\alpha}_{mean}$), posterior median ($\hat{\alpha}_{med}$), 95% CI, and one sided 63% lower credibility bound ($c_{Lower}^{\alpha,0.63}$) for scale parameter *α*.

$\hat{\beta}_{mean}$	$\hat{\beta}_{med}$	$CI_{\beta}^{0.95}=\left(c_{inf}^{\beta,0.95},c_{\sup}^{\beta,0.95}\right)$	$c_{Lower}^{\beta,0.63}$
6.02	6.00	(5.07, 7.37)	5.99

Table 4: Posterior mean ($\hat{\beta}_{mean}$), posterior median ($\hat{\beta}_{med}$), 95% CI, and one-sided 63% lower credibility bound ($c_{Lower}^{\beta,0.63}$) for scale parameter β .

The Bayesian framework gives a shape parameter value of almost 6 (either when referring to posterior median or posterior mean). This means that the available evidence supports the prior knowledge on β. With respect to the scale parameter, the one-sided 63% lower confidence bound $\alpha_{Weib}^{0.63}$ =662 of the Weibayes approach and the-one sided 63% lower credibility bound $c_{Lower}^{\alpha,0.63}$ =921.7 estimated by the

MCMC are quite far from each other (almost 30% of difference). In particular, the result of the Weibayes method is more conservative.

4.2 ***Biased Problem***

The value of the shape parameter considered in this problem is $\hat{\beta}_{Weib} = 0.70$, i.e., far from that used by GE in industrial practice. The aim, in fact, is to propose a comparison of the Weibayes and MCMC approaches, when the a priori knowledge is not supported by the available evidence.

The application of the Weibayes approach when considering a one-sided 63% lower confidence bound on the scale parameter yields

$$\alpha_{Weib}^{0.63} \approx 14007 \tag{27}$$

The MCMC bayesian analysis has been performed with the following prior distributions:

- The generalized improper Jeffery prior distribution of Equation 26, with hyper-parameter $a = 2$
- For the shape parameter, we have chosen a distribution centered on β=0.70, with a probability mass of 0.8 uniformly distributed between [0.5, 0.9] (i.e., symmetrically on β=0.70), a probability mass of 0.1 uniformly distributed in [0, 0.5], whereas the remaining probability mass of 0.1 is uniformly distributed in [0.9, 6]. This choice is justified by the following considerations:
    - To exploit the assumed biased prior knowledge, we have put a large portion of probability on the interval [0.5, 0.9] that encompasses the value $\hat{\beta}_{Weib} = 0.70$.
    - The prior distribution leaves a relatively small portion of probability in the remaining parts of the interval [0,6]. This is consistent with our prior knowledge: we assume that values in [0,6] are plausible values for parameter β.

To draw samples from the posterior distribution $\pi(\alpha,\beta|\boldsymbol{D})$, the ARWMH algorithm described in Section 3 has been run for $M = 5000000$ iterations with variables $\alpha_0, \beta_0, \boldsymbol{\rho}_0$ $\boldsymbol{\Sigma}_0$, and ξ_0 initialized as reported in Table 1. Then, we have applied:

1) a burn-in of 1,000,000 samples.
2) a thinning of 50 samples.

By so doing, the cardinality of the original Markov chains has reduced from 4,000,000 to $\tilde{M} = 80000$ sample points. We have also checked the shape of the trace plots, and applied the two diagnostic methods ESS, and Geweke tests. For brevity, these are not reported. Figures 7 and 8 show the KDE (red dash line) and the EH of $\pi(\alpha|\boldsymbol{D})$ and $\pi(\beta|\boldsymbol{D})$, respectively, whereas the information in the posterior distribution is summarized by in Table 5 and Table 6.

In this biased case study, the estimations of Weibayes and MCMC methods are very different from each other. On the contrary, the estimations of the MCMC approach for the two different a priori settings are not very far.

This seems to suggest that the MCMC method offers more robustness than the Weibayes approach, as the Bayesian framework allows adjusting the initial estimation of β, even if it is not consistent with the gathered evidence. On the contrary, the possibility of adjusting the prior estimation based on the gathered evidence is not offered by the Weibayes method. This may constitute a limitation for the Weibayes method, as it leads to provide wrong results if the value of the shape parameter is not correctly set. On the other side, this disadvantage is counter-balanced by the fact that Weibayes is simpler and faster than MCMC. However, additional research work needs to be carried out both to investigate under which conditions and to which extent Bayesian framework results to be more flexible than Weibayes, and to explore the sensitivity of the accuracy of the results the two methods to the dataset cardinality, information content of the prior knowledge, etc.

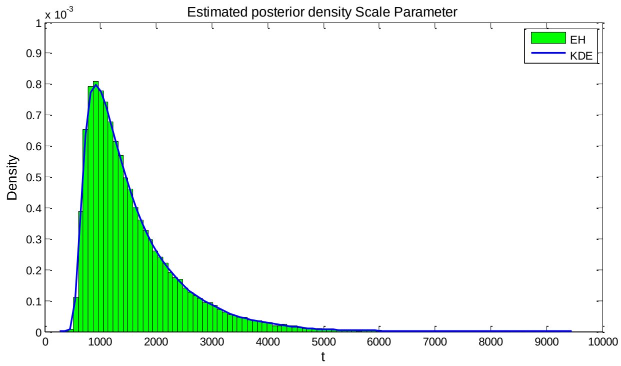

Figure 5: KDE and EH of posterior pdf $\pi(\alpha|\boldsymbol{D})$

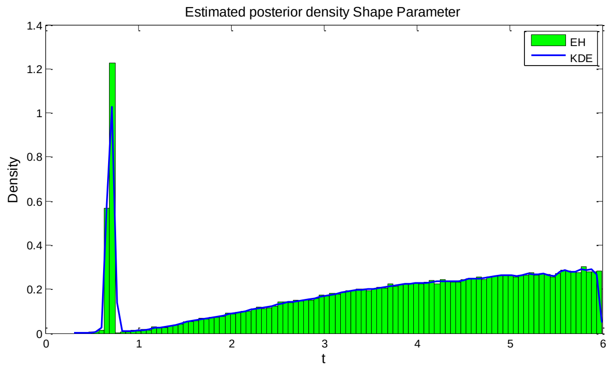

Figure 6: KDE and EH of posterior pdf $\pi(\beta|\boldsymbol{D})$

$\hat{\alpha}_{mean}$	$\hat{\alpha}_{med}$	$CI_{\alpha}^{0.95} = \left(c_{inf}^{\alpha,0.95}, c_{\sup}^{\alpha,0.95}\right)$	$c_{Lower}^{\alpha,0.63}$
1573	1322	(651.47, 3883)	1123

Table 5: Posterior mean ($\hat{\alpha}_{mean}$), posterior median ($\hat{\alpha}_{med}$), 95% CI ($CI_{\alpha}^{0.95}$) and the-one sided 63% lower credibility bound ($c_{Lower}^{\alpha,0.63}$) for scale parameter α

$\hat{\beta}_{mean}$	$\hat{\beta}_{med}$	$CI_{\beta}^{0.95} = \left(c_{inf}^{\beta,0.95}, c_{\sup}^{\beta,0.95}\right)$	$c_{Lower}^{\beta,0.63}$
3.7967	4.0761	(0.6901, 5.9106)	3.4740

Table 6: Posterior mean ($\hat{\beta}_{mean}$), posterior median ($\hat{\beta}_{med}$), 95% CI ($CI_{\beta}^{0.95}$) and the-one sided 63% lower credibility bound ($c_{Lower}^{\beta,0.63}$)for scale parameter β

5 CONCLUSION

In this work, we have considered the realistic problem in reliability analysis of Weibull parameter estimations when the collected field data contain right-censored times only. The Weibayes method and the Bayesian approach have been applied to a real industrial case study concerning data of GT forced outages due to a mechanical failure of a component. This application has shown that the two methods provide the same value for the shape parameter, but different values for the scale parameters.

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